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MONDAY, MAY 9TH, 1853.

THOMAS ROMNEY ROBINSON, D. D., PRESIDENT,
in the Chair.

THE Rev. Dr. Todd presented the following donations to the Museum:

1. From William Acheson, Esq., copies of two ancient brooches,—one, Irish, found at Roscrea, and now in the cabinet of George Petrie, LL. D.; the other, Scotch, found in the parish of West Kilbride, Ayrshire.

2. From the Very Rev. Richard Butler, a steel arrow-head, found on Cromwell's Hill, close to Towneley, on the Boyne.

The Secretary reported that the conditions upon which the Academy's consent to exhibit their Museum, as amended by the vote of the Academy at its last meeting, having been communicated to the General Executive Committee of the Exhibition, were approved of, and signed by their Chairman.

The Rev. Dr. Graves read a paper on the properties of the functions of two variables employed by him in the interpretation of his theory of triplets.

"In interpreting my theory of algebraic triplets I was led to the conception of a calculus, whose formulæ bear a close resemblance to those of trigonometry. In this latter calculus there are two *primitive* functions of a single variable, named its *sine* and *cosine*, between which the equation

$$\sin^2 \theta + \cos^2 \theta = 1$$

holds good, whatever be the value of the variable θ. Besides these primitive functions, there are other *derived* ones, such as the tangent, secant, cotangent, cosecant, &c., which may be expressed by means of the primitive ones; and a number of useful and elegant formulæ may be established, connecting the different trigonometric functions of the same variable, or of variables related to each other in particular ways.

"In the new calculus we have to consider three primitive functions, each of two variables, ϕ and χ, to which I have ventured to give the names of cotresine (ϕ, χ), tresine (ϕ, χ), and tresine (χ, ϕ); and I have found that the two functions obtained by dividing the two latter by the first possess properties analogous to those of the trigonometric tangent, and are sufficiently remarkable to entitle them to a particular designation. I therefore propose to call them *tritangents*.

"In the present paper I mean to give a few of the formulæ which result from a comparison of the functions already noticed.

"Employing the exponential development, putting a for $\frac{-1 \pm \sqrt{(-3)}}{2}$, the cube root of $+1$, and writing $n!$ for the product $1.2.3 \ldots n$, we find

$$e^{a\phi} = \lambda + a\mu + a^2\nu;$$

where

$$\lambda = 1 + \frac{\phi^3}{3!} + \frac{\phi^6}{6!} + \&c.$$

$$\mu = \phi + \frac{\phi^4}{4!} + \frac{\phi^7}{7!} + \&c.$$

$$\nu = \frac{\phi^2}{2!} + \frac{\phi^5}{5!} + \frac{\phi^8}{8!} + \&c.$$

Again,

$$e^{a^2\chi} = \lambda_1 + a^2\mu_1 + a\nu_1,$$

where λ_1, μ_1, ν_1 are the same functions of χ that λ, μ, ν are of ϕ. Hence $e^{a\phi + a^2\chi} = \text{cotr}\,(\phi, \chi) + a\,\text{tres}\,(\phi, \chi) + a^2\,\text{tres}\,(\chi, \phi)$ if we agree to put as definitions,

$$\begin{aligned} \text{cotr}\,(\phi, \chi) &= \lambda\lambda_1 + \mu\mu_1 + \nu\nu_1, \\ \text{tres}\,(\phi, \chi) &= \lambda\nu_1 + \mu\lambda_1 + \nu\mu_1, \\ \text{tres}\,(\chi, \phi) &= \lambda\mu_1 + \mu\nu_1 + \nu\lambda_1. \end{aligned} \qquad (1)$$

“ The exponential values of the sine and cosine are most useful in all parts of analysis. Analogous values may be assigned to the cotresine or tresine of two variables. Since

$$\begin{aligned} e^{a\phi+a^2\chi} &= \operatorname{cotr}(\phi, \chi) + a \operatorname{tres}(\phi, \chi) + a^2 \operatorname{tres}(\chi, \phi), \\ e^{a^2\phi+a\chi} &= \operatorname{cotr}(\phi, \chi) + a^2 \operatorname{tres}(\phi, \chi) + a \operatorname{tres}(\chi, \phi), \\ e^{\phi+\chi} &= \operatorname{cotr}(\phi, \chi) + \operatorname{tres}(\phi, \chi) + \operatorname{tres}(\chi, \phi), \end{aligned} \qquad (2)$$

Adding these equations, and dividing by 3, we get

$$\operatorname{cotr}(\phi, \chi) = \tfrac{1}{3}\left\{e^{a\phi+a^2\chi} + e^{a^2\phi+a\chi} + e^{\phi+\chi}\right\}.$$

In like manner we should find

$$\operatorname{tres}(\phi, \chi) = \tfrac{1}{3}\left\{a^2 e^{a\phi+a^2\chi} + a e^{a^2\phi+a\chi} + e^{\phi+\chi}\right\}, \qquad (3)$$

and
$$\operatorname{tres}(\chi, \phi) = \tfrac{1}{3}\left\{a e^{a\phi+a^2\phi} + a^2 e^{a^2\phi+a\chi} + e^{\phi+\chi}\right\}.$$

Multiplying together the first and second of equations (2), we get

$$\begin{aligned} e^{-\phi-\chi} = {} & \operatorname{cotr}^2(\phi, \chi) + \operatorname{tres}^2(\phi, \chi) + \operatorname{tres}^2(\chi, \phi) - \operatorname{tres}(\phi, \chi) \operatorname{tres}(\chi, \phi) \\ & - \operatorname{tres}(\chi, \phi) \operatorname{cotr}(\phi, \chi) - \operatorname{cotr}(\phi, \chi) \operatorname{tres}(\phi, \chi); \end{aligned}$$

and multiplying this again by the third, we find

$$\begin{aligned} & \operatorname{cotr}^3(\phi, \chi) + \operatorname{tres}^3(\phi, \chi) + \operatorname{tres}^3(\chi, \phi) \\ & \qquad - 3 \operatorname{cotr}(\phi, \chi) \operatorname{tres}(\phi, \chi) \operatorname{tres}(\chi, \phi) = 1. \end{aligned}$$

This equation, which holds good whatever be the values of the variables ϕ and χ, corresponds in this calculus to the well-known relation between the sine and cosine.

“ In trigonometry we have

$$\cos(-\theta) = \cos\theta, \text{ and } \sin(-\theta) = -\sin\theta.$$

The corresponding formulæ in this calculus are the following:

$$\begin{aligned} \operatorname{cotr}(a\phi, a^2\chi) &= \operatorname{cotr}(\phi, \chi), \\ \operatorname{tres}(a\phi, a^2\chi) &= a \operatorname{tres}(\phi, \chi), \\ \operatorname{tres}(a^2\chi, a\phi) &= a^2 \operatorname{tres}(\chi, \phi). \end{aligned} \qquad (4)$$

“ The relations between the cotresine or tresine of two variables, and the same functions of these variables with their signs changed, are not quite so simple. They are, however, easily established; and instances frequently arise in which it is necessary to avail ourselves of them.

" Changing the signs of ϕ and χ in the first of formulæ (3), we have

$$3 \text{ cotres}(-\phi, -\chi) = \frac{1}{\{\text{cotr}(\phi,\chi) + a\,\text{tres}(\phi,\chi) + a^2\,\text{tres}(\chi,\phi)\}} + \frac{1}{\text{cotr}(\phi,\chi) + a^2\,\text{tres}(\phi,\chi) + a\,\text{tres}(\chi,\phi)} + \frac{1}{\text{cotr}(\phi,\chi) + \text{tres}(\phi,\chi) + \text{tres}(\chi,\phi)}$$

and on adding these fractions, we get

$$\text{cotr}(-\phi, -\chi) = \text{cotr}^2(\phi,\chi) - \text{tres}(\phi,\chi)\,\text{tres}(\chi,\phi).$$

By a similar process we should obtain

$$\begin{aligned} \text{tres}(-\phi, -\chi) &= \text{tres}^2(\chi,\phi) - \text{cotr}(\phi,\chi)\,\text{tres}(\phi,\chi), \\ \text{tres}(-\chi, -\phi) &= \text{tres}^2(\phi,\chi) - \text{cotr}(\phi,\chi)\,\text{tres}(\chi,\phi). \end{aligned}$$

These last expressions are particularly useful in geometrical applications of this theory.

" The known formulæ for the sine and cosine of the sum of two arcs may be most readily derived from the equation

$$e^{\theta\sqrt{-1}} = \cos\theta + \sin\theta\,.\,\sqrt{-1}.$$

In like manner we may obtain formulæ for the tresines and cotresine of $\phi + h$ and $\chi + k$ from the equations (2). Thus, if ι denote a symbol of distributive operation such that $\iota^3 = 1$, whilst ι, ι^2, and 1 are absolutely heterogeneous, we shall have

$$\begin{aligned} e^{\iota\phi} &= \lambda + \iota\mu + \iota^2\nu, \\ e^{\iota^2\chi} &= \lambda' + \iota^2\mu' + \iota\nu'; \end{aligned}$$

whence

$$e^{\iota\phi+\iota^2\chi} = \text{cotr}(\phi,\chi) + \iota\,\text{tres}(\phi,\chi) + \iota^2\,\text{tres}(\chi,\phi).$$

In like manner,

$$e^{\iota h+\iota^2 k} = \text{cotr}(h,k) + \iota\,\text{tres}(h,k) + \iota^2\,\text{tres}(k,h).$$

Multiplying the two latter equations together, we have

$$
\begin{aligned}
e^{\iota(\phi+h)+\iota^2(\chi+k)} = \operatorname{cotr}(\phi, \chi)\operatorname{cotr}(h,k) &+ \operatorname{tres}(\phi, \chi)\operatorname{tres}(k,h) \\
&+ \operatorname{tres}(\chi, \phi)\operatorname{tres}(h, k) \\
+ \iota\,\{\operatorname{tres}(\chi, \phi)\operatorname{tres}(k, h) &+ \operatorname{cotr}(\phi, \chi)\operatorname{tres}(h, k) \\
&+ \operatorname{tres}(\phi, \chi)\operatorname{cotr}(h, k)\} \\
+ \iota^2\,\{\operatorname{tres}(\phi, \chi)\operatorname{tres}(h, k) &+ \operatorname{tres}(\chi, \phi)\operatorname{cotr}(h,k) \\
&+ \operatorname{cotr}(\phi, \chi)\operatorname{tres}(k, h)\}.
\end{aligned}
$$

Now the left-hand member in this equation being also equal to

$$\operatorname{cotr}(\phi + h,\ \chi + k) + \iota\operatorname{tres}(\phi + h,\ \chi + k) + \iota^2\operatorname{tres}(\chi + k,\ \phi + h),$$

we may compare the *similar* parts of the two expressions, and thus get at once the three formulæ of which we were in search, viz.:

$$
\begin{aligned}
\operatorname{cotr}(\phi + h,\ \chi + k) &= \operatorname{cotr}(\phi, \chi)\operatorname{cotr}(h, k) \\
&\quad + \operatorname{tres}(\phi, \chi)\operatorname{tres}(k, h) + \operatorname{tres}(\chi, \phi)\operatorname{tres}(h, k), \\
\operatorname{tres}(\phi + h,\ \chi + k) &= \operatorname{tres}(\chi, \phi)\operatorname{tres}(k, h) \\
&\quad + \operatorname{cotr}(\phi, \chi)\operatorname{tres}(h, k) + \operatorname{tres}(\phi, \chi)\operatorname{cotr}(h, k), \\
\operatorname{tres}(\chi + k,\ \phi + h) &= \operatorname{tres}(\phi, \chi)\operatorname{tres}(h, k) \\
&\quad + \operatorname{tres}(\chi, \phi)\operatorname{cotr}(h, k) + \operatorname{cotr}(\phi, \chi)\operatorname{tres}(k, h).
\end{aligned}
$$

From these equations, combined with (4), we obtain

$$
\begin{aligned}
\operatorname{cotr}(\phi + h,\ \chi + k) &+ \operatorname{cotr}(\phi + ah,\ \chi + a^2k) \\
&+ \operatorname{cotr}(\phi + a^2h,\ \chi + ak) = 3\operatorname{cotr}(\phi, \chi)\operatorname{cotr}(h, k),
\end{aligned}
$$

which is obviously analogous to the formula

$$\cos(\theta + h) + \cos(\theta - h) = 2\cos\theta\cos h;$$

and we might obtain similar formulæ for the tresines.

" In many investigations great convenience arises from the peculiar way in which the functions of our new calculus are affected by differentiation.

" From formulæ (1) or (3) we obtain the following:

$$\frac{d\,\text{cotr}(\phi, \chi)}{d\phi} = \text{tres}\,(\chi, \phi), \qquad \frac{d\,\text{cotr}\,(\phi, \chi)}{d\chi} = \text{tres}\,(\phi, \chi),$$

$$\frac{d\,\text{tres}\,(\phi, \chi)}{d\phi} = \text{cotr}\,(\phi, \chi), \qquad \frac{d\,\text{tres}\,(\phi, \chi)}{d\chi} = \text{tres}\,(\chi, \phi),$$

$$\frac{d\,\text{tres}\,(\chi, \phi)}{d\phi} = \text{tres}\,(\phi, \chi), \qquad \frac{d\,\text{tres}\,(\chi, \phi)}{d\chi} = \text{cotr}\,(\phi, \chi).$$

It appears, then, that the symbol $\frac{d}{d\phi}$ operating upon any one of the functions cotr (ϕ, χ), tres (ϕ, χ), tres (χ, ϕ), changes it into the preceding function in that cycle; whilst the symbol $\frac{d}{d\phi}$ changes any one of these functions into the succeeding one in the same cycle. It follows, therefore, that $\frac{d^2}{d\phi d\theta}$ is inoperative upon each of these three functions. These results are analogous to those which we are familiar with in trigonometry, where we find sines and cosines reproduced by differentiation.

"In discussing the properties of the surface whose equation in rectangular co-ordinates is

$$x^3 + y^3 + z^3 - 3\,xyz = 1,$$

as the co-ordinates x, y, z are equal respectively to cotr (ϕ, χ), tres (ϕ, χ), tres (χ, ϕ), it will be convenient to denote

$$\text{cotr}\,(-\phi, -\chi),\ \text{tres}\,(-\phi, -\chi),\ \text{tres}\,(-\chi, -\phi) \text{ by } \bar{x}, \bar{y}, \bar{z}.$$

Then, as

$$x^2 - yz = \bar{x}, \quad y^2 - zx = \bar{z}, \quad z^2 - xy = \bar{y},$$

the equation of the tangent plane at the point (x', y', z') will be

$$\bar{x}'x + \bar{z}'y + \bar{y}'z = 1.$$

Hence dS, the element of the surface, is expressed by

$$\frac{(\bar{x}^2 + \bar{y} + \bar{z}^2)^{\frac{1}{2}}\,dydz}{\bar{x}};$$

and as the perpendicular from the origin upon the tangent plane is equal to

$$(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{-\frac{1}{2}},$$

we find the volume of the small pyramid, which has the origin for its vertex and dS for its base, to be equal to

$$\tfrac{1}{3}\frac{dydz}{\bar{x}}.$$

But again, as

$$dx = zd\phi + yd\chi,$$
$$dy = xd\phi + zd\chi,$$
$$dz = yd\phi + xd\chi,$$

the element $dydz$ must be replaced by $(x^2 - yz)\,d\phi d\chi$ under the double sign of integration. Hence

$$\tfrac{1}{3}\iint\frac{dydz}{\bar{x}} = \tfrac{1}{3}\iint d\phi d\chi.$$

Thus it appears that the sector, generated by a radius vector drawn from the origin, and having for its base the portion of the surface bounded by the lines $\phi = 0$, $\chi = 0$, $\phi = \phi'$, $\chi = \chi'$, may be represented by the product $\frac{1}{3}\phi'\chi'$.

"It has been observed that the functions obtained by dividing the two tresines by the cotresine possess properties analogous to those of the trigonometric *tangent*. A remarkable instance to this effect may be adduced in connexion with the preceding theorem.

"Let us put

$$\text{trit}\,(\phi, \chi) = \frac{y}{x} = \eta, \text{ and } \text{trit}\,(\chi, \phi) = \frac{z}{x} = \zeta;$$

then

$$d\eta = \frac{xdy - ydx}{x^2}, \text{ and } d\zeta = \frac{xdz - zdx}{x^2};$$

or

$$d\eta = \frac{(x^2 - yz)\,d\phi - (y^2 - zx)\,d\chi}{x^2} = \frac{\bar{x}d\phi - \bar{z}d\chi}{x^2},$$

and